Subject: Leaving Certificate Maths

Teacher: Mr Murphy

Lesson 16: Calculus II

16.1 Learning Intentions

After this week's lesson you will be able to;

- Differentiate Trigonometric functions
- Logarithmic & Exponential Functions
- Identify a function and its derivative from a graph.
- Find the 2^{nd} Derivative of a function.
- Find the maximum and minimum points of a function.

16.2 Specification

5.2 Calculus	– find first and second derivatives of linear, quadratic and cubic functions by rule – associate derivatives with slopes and tangent lines – apply differentiation to • rates of change • maxima and minima • curve sketching	– differentiate linear and quadratic functions from first principles – differentiate the following functions • polynomial • exponential • trigonometric • rational powers • inverse functions • logarithms – find the derivatives of sums, differences, products, quotients and compositions of functions of the above form – apply the differentiation of above functions to solve problems

16.3 Chief Examiner's Report

B	8	26·6	53	8	Functions/rates of change
B	9	30·7	61	6	Functions/trigonometry/calculus

16.4 Trigonometric Functions

These functions include any of the three trigonometric ratios:

Sin
Cos
Tan

$f(x) = \sin x \; f'(x) = \cos x$

$g(\theta) = \cos\theta \; g'(\theta) = -\sin\theta$

$y = \tan\theta \quad \frac{dy}{d\theta} = \sec^2\theta$

This is a more complicated example:

$f(x) = Cos^2\theta$

$f(x) = (Cos\,\theta)^2$.....this is a chain rule

$f'(x) = 2(Cos\,\theta)^{2-1}.(-Sin\,x)$

$f'(x) = -2Sin\,\theta.Cos\,\theta$

Questions:

1) $f(x) = 3Sin\,x$

2) $g(x) = x^2.Cos\,x$

3) $h(x) = \frac{4x}{Sin\,x}$

Inverse Trigonometric Functions:

Remembering that these functions are identified by an index (power) of minus one at the end of the trigonometric function, for example:

$Sin^{-1}\,x$

Below write down the derivatives that correspond to the functions from the video:

Function Derivative

$Sin^{-1}\frac{x}{a}$

$Cos^{-1}\frac{x}{a}$

$Tan^{-1}\frac{x}{a}$

Sample Questions:

$f(x) = Sin^{-1}\frac{x}{3}$

$g(x) = \frac{Sin^{-1} x}{x^2}$

$g(x) = Sin^{-1} x^2$

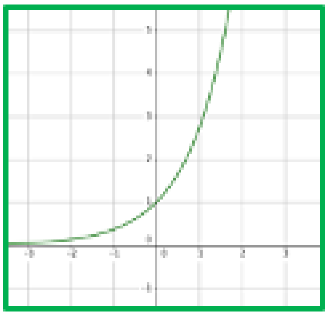
16.5 e and ln Functions

A function containing e^x can be described as the exponential function

$f(x) = e^x$

Then

$$f'(x) = e^x$$



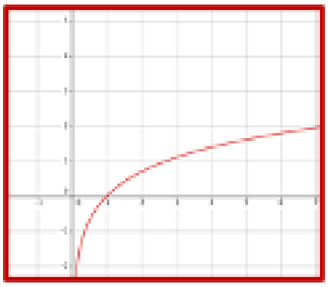
$f(x) = e^x$

A function with $\log_e x$ can be described as $\ln x$

$g(x) = \ln x$

Then

$g^{'}(x) = \frac{1}{x}$



$g(x) = \ln x$

These are the main forms of the two above functions, however there are two more cases to be aware of:

1st Variation (copy down from video):

$f(x) = e^{3x}$

2nd Variation:

Also be careful if x is the exponent. In this case the following rule applies:

If $f(x) = 2^x$

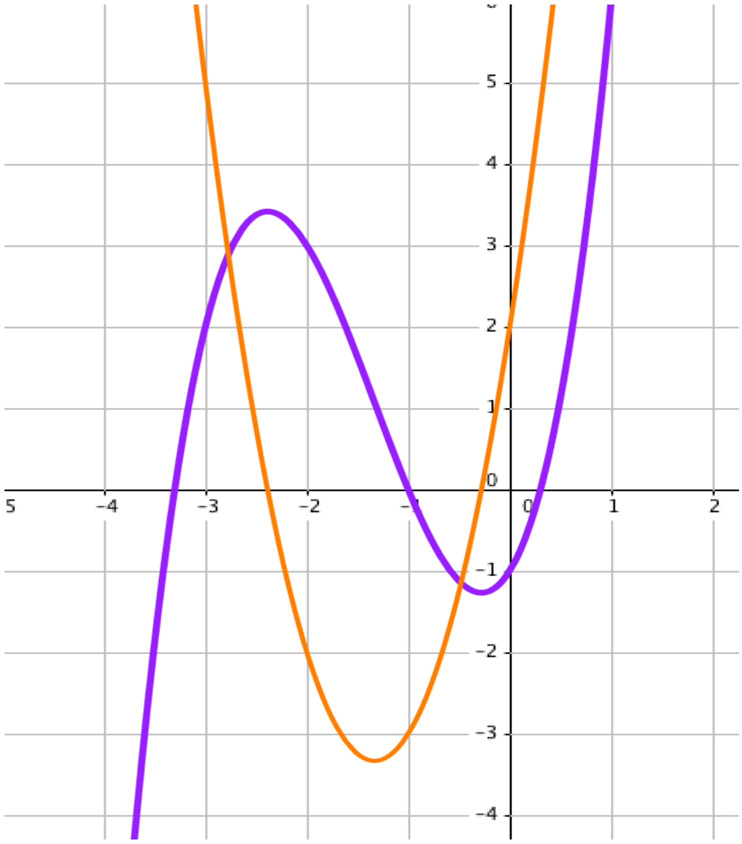
Then

$f^{'}(x) = 2^x(\ln 2)$

16.6 Functions and their derivatives

We can sometimes be required to identify a function and its derivative based on the graphs of the function and slope functions.

Cubic	$x^3 + 4x^2 + 2x - 1$
Quadratic	$3x^2 + 8x + 2$
Linear	$6x + 8$
Constant	6



Using the video add in the key pieces of information, i.e. the link between the two graphs, which is the derivative etc.

16.7 Second Derivatives

The second derivative is the result of differentiating a function twice. We can use the following notation to denote a second derivative:

$$f''(x) \text{ or } \frac{d^2y}{dx^2}$$

Below we look at an application of the concept of differentiation and more importantly second derivatives.

We can use functions to represent many things. Once such this is the distance (s) travelled by an object over a certain time(t).

$$s(t) = 3t^3 + 4t - 1$$

If we differentiate, we can get the rate of change of the function. A rate of change of distance is better referred to as speed or velocity. So, we can say that:

$$s'(t) = v(t)$$

Therefore:

$$v(t) = 9t^2 + 4$$

Likewise, we can differentiate and get a rate of change of velocity, which is better known as acceleration, so:

$$v'(t) = a(t)$$

Therefore:

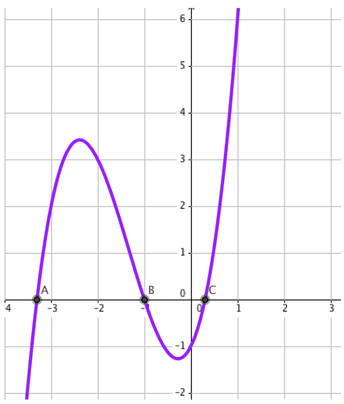
$$a(t) = 18t$$

Using the above functions, we can find the values for velocity or acceleration at any time that is given. To do this all we need to do is sub in a given value for t into any of the functions. Be sure to include the appropriate units in your answer, e.g. m, km, s, ms^{-1}.

16.8 Increasing/Decreasing Functions

On the next page try to identify which parts of the graph you think are increasing and which parts are decreasing as we move from left to right.

The function is a cubic function and has two regions of increasing and one of decreasing.



1) Where is this increasing?

2) Where is this decreasing?

3) Where is it doing neither?

From looking at the video what can we say about the points where the graph is neither increasing nor decreasing?

We refer to these points as **Stationary** or **Turning** points.

At a turning point we can say that:

$$\frac{dy}{dx} \text{ or } f'(x) = 0$$

We can use this information to find the coordinates of the turning points.

In order to find the turning points, we must firstly find the first derivative of the cubic function:

$f(x) = x^3 + 4x^2 + 2x - 1$

$\boldsymbol{f'(x) = 3x^2 + 8x + 2}$

Now we let $\boldsymbol{f'(x) = 0}$ and solve to get the x-values (x-coordinates) of the turning points.

$\boldsymbol{0 = 3x^2 + 8x + 2}$

Using the – quadratic formula we can find two x-values:

$$\boldsymbol{x = -2.9} \text{ and } \boldsymbol{x = -0.28}$$

We then sub these values into the original function to get the corresponding y-coordinates.

$$y = 3.44 \text{ and } y = -1.27$$

So, the coordinates for the two turning points are:

(-2.9, 3.44) **(-0.28 , -1.27)**

Now that we have the values of the turning points, we can be asked to identify the nature of the turning points. In other words, which is the **local maximum** and **local minimum**.

We can use the 2nd derivative to test which point is the local maximum and which is the local minimum.

We firstly establish the 2nd derivative, then we sub in the x-coordinate of the point we are testing. Based on the value of the answer we can determine its nature.

If $\boldsymbol{f''(x) < 0}$ then the that point is the **local maximum.**

If $f''(x) > 0$ then the that point is the **local minimum.**

$f''(x) = 6x + 8$

So now we test the 1st point (-2.9, 3.44):

$f''(-2.9) = 6(-2.9) + 8$

$f''(-2.9) = -9.4$

As 2nd Derivative is negative (< 0) @ x = -2.9, this is the local maximum point.

16.9 Recap of the Learning Intentions

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16.10 Homework Task

The revenue (in euro) can be approximated by the function:

$$r(t) = 22\,500 \cos\left(\frac{\pi}{26}t\right) + 37\,500, \qquad t \geq 0$$

where t is the number of weeks measured from the beginning of July and $\left(\frac{\pi}{26}t\right)$ is in radians.

(a) Find the approximate revenue produced 20 weeks after the beginning of July.
Give your answer correct to the nearest euro.

(b) Find the two values of the time t, within the first 52 weeks, when the revenue is approximately €26 250.

(d) Use calculus to show that the revenue is increasing 30 weeks after the beginning of July.

16.11 Solutions to 15.11

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from first principles.

$f(x) = 2x^2 - 3x - 6$

$f(x+h) = 2(x+h)^2 - 3(x+h) - 6$

$f(x+h) = 2(x^2 + 2xh + h^2) - 3(x+h) - 6$

$f(x+h) = 2x^2 + 4xh + 2h^2 - 3x - 3h - 6$

$f(x+h) - f(x) = 2x^2 + 4xh + 2h^2 - 3x - 3h - 6 - 2x^2 + 3x + 6$

$f(x+h) - f(x) = 4xh + 2h^2 - 3h$

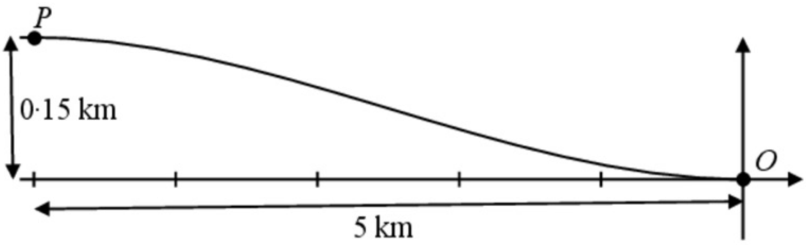
$$\frac{f(x+h) - f(x)}{h} = \frac{4xh + 2h^2 - 3h}{h}$$

$$\frac{f(x+h) - f(x)}{h} = 4x + 2h - 3$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 4x - 3$$

Question: **50 marks**

A plane is flying horizontally at P at a height of 150 m above level ground when it begins its descent. P is 5 km, horizontally, from the point of touchdown O. The plane lands horizontally at O.



Taking O as the origin, $(x, \ f(x))$ approximately describes the path of the plane's descent where $f(x) = 0 \cdot 0024x^3 + 0 \cdot 018x^2 + cx + d, \ -5 \le x \le 0$, and both x and $f(x)$ are measured in km.

(b) **(i)** Find the value of $f'(x)$, the derivative of $f(x)$, when $x = -4$.

$f(x) = 0.0024x^3 + 0.018x^2$

$f'(x) = 0.0072x^2 + 0.036x$

$f'(-4) = 0.0072(-4)^2 + 0.036(-4)$

$f^{'(-4)} = -0.0288$